

# Latency.nb

This is a description of how latency in a [open | closed] loop controller maps to wavefront rms error

The Taylor frozen-flow atmosphere has the power spectrum

$$S_{\phi}[f] := 0.0770491 (r_0 / v)^{-5/3} f^{-8/3}$$

radians<sup>2</sup>/Hz. To convert to nanometers, multiply by  $\lambda/2\pi$ . For a general closed loop control law,  $H_{CL}[f]$ , the residual power spectrum is

$$S_e[f] := \|H_{OLC}[f]\|^2 S_{\phi}[f]$$

In the case of open-loop control, the controller process consists of subtracting an estimated version of the wavefront from the current wavefront. The estimate is formed after integrating the wavefront sensor for a cycle time, then waiting one frame cycle for the WFS camera readout and then a compute delay to calculate the estimate. Finally the estimated wavefront is held on the actuators of the DM for one cycle time. There is no feedback. In the Fourier domain, the process is described by

$$H_{OLC}[f] = 1 - \left( \frac{1 - e^{-i2\pi f T}}{-i2\pi f T} \right) e^{-i2\pi f (T + \tau_c)} \left( \frac{1 - e^{-i2\pi f T}}{-i2\pi f T} \right)$$

where T is the sample time and  $\tau_c$  is the compute delay.

In the case of closed-loop control, the controller has a feedback loop. In the case of integral feedback with a gain,  $\gamma$ , the transfer function is

$$H_{CLC}[f] = \frac{1}{1 + \frac{\gamma}{1 - e^{-i2\pi f T}} \left( \frac{1 - e^{-i2\pi f T}}{-i2\pi f T} \right) e^{-i2\pi f (T + \tau_c)} \left( \frac{1 - e^{-i2\pi f T}}{-i2\pi f T} \right)}$$

The residual wavefront variance is determined by integrating  $S_e[f]$  up to the Nyquist frequency :

$$\sigma_{BW}^2 = 2 \int_0^{\frac{1}{2T}} S_e[f] df$$

This ignores spectral contributions beyond Nyquist, which we'll categorize as "aliasing error" rather than bandwidth error. This reasonable to do for two reasons: a) the sampling time is usually chosen by design so that out-of-band contributions are negligible, b) the Hartmann subapertures actually impose a low-pass filter on the disturbance entering the controller, at a cutoff of  $v/d$  Hz where  $v$  is the wind velocity and  $d$  is the subaperture size, and, again, the sampling frequency  $1/T$  is by design chosen much larger than  $v/d$ . In essence, this is saying that aliasing error due to the finite subaperture size has already counted in all the aliasing, and we don't have to re-count it in the bandwidth error calculation.

Through a change of variables, it is possible to factor out the relevant scale factors and do the integration numerically:

$$\sigma_{BW}^2 = (v T / r_0)^{5/3} \zeta$$

where  $\zeta$  is a constant that depends on the case of open or closed loop, the delay parameter  $\alpha = \tau_c/T$ , and in the case of closed loop, the closed loop gain,  $\gamma$ .

## ■ Comparison to Greenwood Frequency

The Greenwood Frequency is a parameter relating the rms closed loop residual in an ideal continuous-time integral feedback controller to the atmospheric conditions and the closed-loop -3db rejection frequency,  $f_c$ .

$$\sigma_{BW}^2 = (f_g / f_c)^{5/3}$$

$$f_g = 0.135 / \tau_0 = \frac{0.135}{0.314} \times \frac{v}{r_0}$$

$$\sigma_{BW}^2 = \left( \frac{0.135}{0.314} \right)^{5/3} \left( \frac{1}{f_c T} \right)^{5/3} \left( \frac{v T}{r_0} \right)^{5/3} = \zeta_g \left( \frac{v T}{r_0} \right)^{5/3}$$

For the case of discrete time systems with delays, it is difficult to get a bandwidth of rejection that is more than about 1/10 of the sample frequency. The discussion below shows that, with a compute delay combined with the camera readout, WFS integration, and DM hold times, the bandwidth is about 0.05/T when the compute delay is one sample and 0.025/T when the compute delay is two samples. Substituting this in to the above formula gives approximate values for  $\zeta_g$ :

$$\text{In[179]:= } \zeta_g = \left\{ \left( \frac{0.135}{0.314} \right)^{5/3} \left( \frac{1}{0.05} \right)^{5/3}, \left( \frac{0.135}{0.314} \right)^{5/3} \left( \frac{1}{0.025} \right)^{5/3} \right\} // \mathbf{N}$$

Out[179]= {36.0903, 114.579}

### ■ Comparison to pure time-lag ( $\tau_0$ )

In a simple open-loop lag model, the correction is applied  $\tau_d$  seconds delayed. The residual is then

$$\sigma_{BW}^2 = 6.88 \left( \frac{v \tau_d}{r_0} \right)^{5/3} = \zeta_0 \left( \frac{v T}{r_0} \right)^{5/3}$$

We can approximate the WFS stare time and DM hold time as latency delays of 1/2 a sample period each. The approximation is justified by noting that this is the "average" age of the data in each case (the calculations in the next section vindicate this). Add to this WFS readout time and compute delays of either one or two sample periods and the equivalent values for  $\zeta_0$  are:

$$\text{In[178]:= } \zeta_0 = 6.88 \times \{3, 4\}^{5/3}$$

Out[178]= {42.9329, 69.3461}

### ■ Discrete time systems

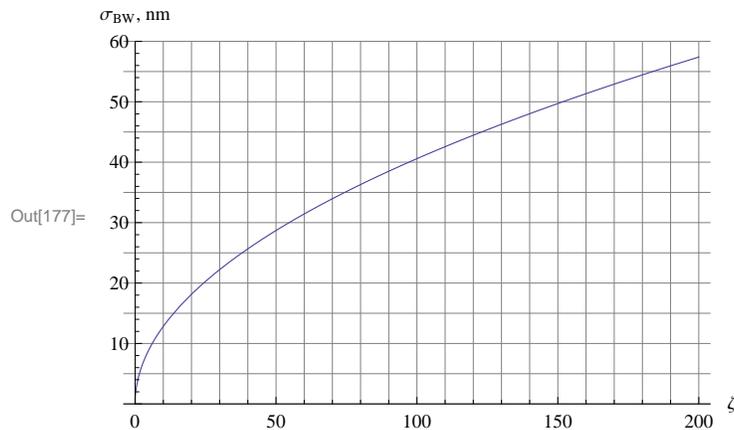
In the next section we show that the actual values are considerably larger than this, owing to the fact that the delays actually cause overshoot (gains  $> 1$  at some high frequency band) in the rejection curves. For open-loop the values are:  $\zeta = 84$  and 137 for one and two T compute delays respectively. For closed-loop (with optimally tuned feedback gain) the values are:  $\zeta = 98$  and 172 for one and two T compute delays respectively.

	$\alpha = 1$	$\alpha = 2$
Greenwood, $f_g$	36	115
Time - lag, $\tau_0$	43	69
Closed loop	98	172
Open loop	84	137

**Table** of normalized residual variances,  $\zeta$ , as computed by various methods for one and two samples of compute delay.

We can substitute the Keck NGAO nominal values for  $v$  and  $r_0$  to translate these normalized residuals into nanometers of wavefront error. For the nominal NGAO design conditions,  $v = 9$  m/sec,  $r_0 = 16$  cm at  $\lambda = 0.5 \mu\text{m}$ , and  $T = 0.5$  ms this yields

```
In[177]:= Plot[(500 / (2 π)) × (9 × 0.0005 / 0.16)5/6 × √ζ, {ζ, 0, 200}, AxesLabel → {"ζ", "σBW, nm"},
PlotRange → {0, 60}, GridLines → {Range[0, 200, 10], Range[0, 60, 5]}]
```



**Plot** of rms residual wavefront due to control loop following error as a function of the constant parameter  $\zeta$ . All the values mentioned in the text lie on this graph.

In all cases considered, including discrete-time open or closed loop with 2 samples of compute delay,  $\zeta < 200$  and so  $\sigma_{BW} < 60$  nm.

```
In[207]:= Style[Grid[Transpose[
  {"σBW(nm)", "Greenwood", "Time-lag", "Closed loop", "Open Loop"}, Join[{"α=1"},
  Table[(500 / (2 π)) × (9 × 0.0005 / 0.16)5/6 × √ζ, {ζ, {36, 43, 98, 84}}] // Round],
  Join[{"α=2"}, Table[(500 / (2 π)) × (9 × 0.0005 / 0.16)5/6 × √ζ, {ζ, {115, 69, 172, 137}}] //
  Round]]], Frame → All], FontFamily → "Times"]
```

Out[207]=

$\sigma_{BW}(\text{nm})$	$\alpha=1$	$\alpha=2$
Greenwood	24	44
Time-lag	27	34
Closed loop	40	53
Open Loop	37	48

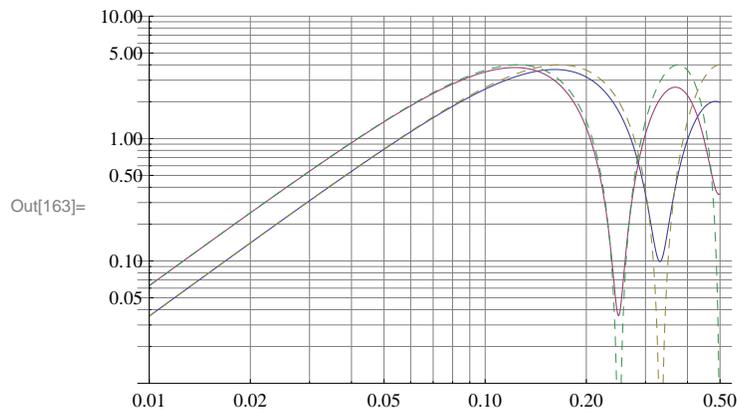
**Table** of residual wavefront errors for the NGAO nominal conditions. Units are nanometers, rms.

## Appendix: Rejection performance of discrete-time control laws

In this section we calculate  $\zeta$  for the discrete-time control laws. As mentioned in the main text,  $\zeta$  is considerably higher in the case of a real system with delays than it is in the simplified models.

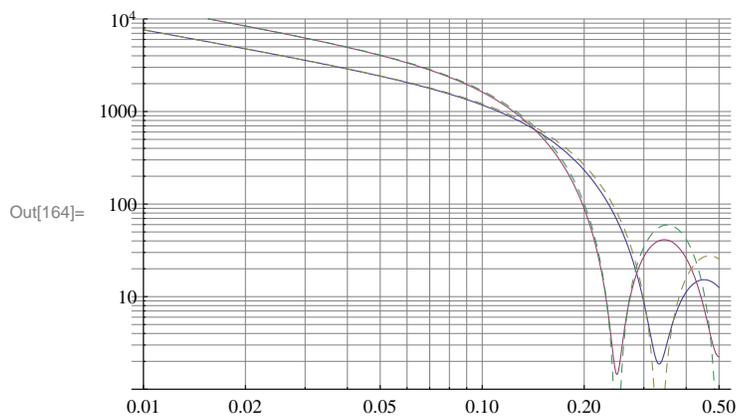
## ■ Open loop transfer function curves

```
In[163]:= LogLogPlot [ {
  Abs [ 1 - e-i 2 π α f (  $\frac{1 - e^{-i 2 \pi f}}$  )2 ]2 /. α → 2,
  Abs [ 1 - e-i 2 π α f (  $\frac{1 - e^{-i 2 \pi f}}$  )2 ]2 /. α → 3,
  Abs [ 1 - e-i 2 π α f ]2 /. α → 3,
  Abs [ 1 - e-i 2 π α f ]2 /. α → 4 },
  { f, .01, 1 / 2 }, GridLines → Automatic,
  PlotStyle → { Automatic, Automatic, Dashed, Dashed }, PlotRange → { .01, 10 } ]
```



Open loop transfer functions for 1 and 2 sample-time compute delays (increasing crossover frequency with decreasing compute delay). The dashed curves represent the "average age of data" approximation where we substitute the WFS stare and DM hold transfer functions each with 1/2 sample time delays.

```
In[164]:= LogLogPlot[{
  Abs[1 - e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2]^2 f^{-8/3} /. \alpha \to 2,
  Abs[1 - e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2]^2 f^{-8/3} /. \alpha \to 3,
  Abs[1 - e^{-i 2 \pi \alpha f}]^2 f^{-8/3} /. \alpha \to 3,
  Abs[1 - e^{-i 2 \pi \alpha f}]^2 f^{-8/3} /. \alpha \to 4},
{f, .01, 1 / 2},
GridLines -> Automatic,
PlotRange -> {1, 10 000},
PlotStyle -> {Automatic, Automatic, Dashed, Dashed}]
```



Open loop residual spectra for 1 and 2 sample-time compute delays (decreasing residual with decreasing compute delay). The dashed curves represent the "average age of data" approximation where we substitute the WFS stare and DM hold transfer functions each with 1/2 sample time delays.

```
In[165]:= 0.0770491 \times 2 \left( NIntegrate[{
  Abs[1 - e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2]^2 f^{-8/3} /. \alpha \to 2,
  Abs[1 - e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2]^2 f^{-8/3} /. \alpha \to 3,
  Abs[1 - e^{-i 2 \pi \alpha f}]^2 f^{-8/3} /. \alpha \to 3,
  Abs[1 - e^{-i 2 \pi \alpha f}]^2 f^{-8/3} /. \alpha \to 4},
{f, 10^{-6}, 1 / 2}] +
{12 \pi^2 (1 + \alpha)^2 f_1^{1/3} /. \alpha \to 2, 12 \pi^2 (1 + \alpha)^2 f_1^{1/3} /. \alpha \to 3,
12 \pi^2 (1 + \alpha)^2 f_1^{1/3} /. \alpha \to 2, 12 \pi^2 (1 + \alpha)^2 f_1^{1/3} /. \alpha \to 3} /. f_1 \to 10^{-6} // Re
```

```
Out[165]= {83.9864, 136.788, 85.302, 138.198}
```

$$\text{In[114]:= Series} \left[ \left( 1 - e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2 \right) \left( 1 - e^{i 2 \pi \alpha f} \left( \frac{1 - e^{i 2 \pi f}}{i 2 \pi f} \right)^2 \right) f^{-8/3}, \{f, 0, 3\} \right]$$

$$\text{Out[114]=} \frac{4 \pi^2 (1 + \alpha)^2}{f^{2/3}} - \frac{1}{9} \left( \pi^4 (23 + 72 \alpha + 84 \alpha^2 + 48 \alpha^3 + 12 \alpha^4) \right) f^{4/3} + O[f]^{10/3}$$

$$\text{In[118]:= Integrate} \left[ \frac{4 \pi^2 (1 + \alpha)^2}{f^{2/3}}, \{f, 0, f_1\} \right]$$

$$\text{Out[118]=} 12 \pi^2 (1 + \alpha)^2 f_1^{1/3}$$

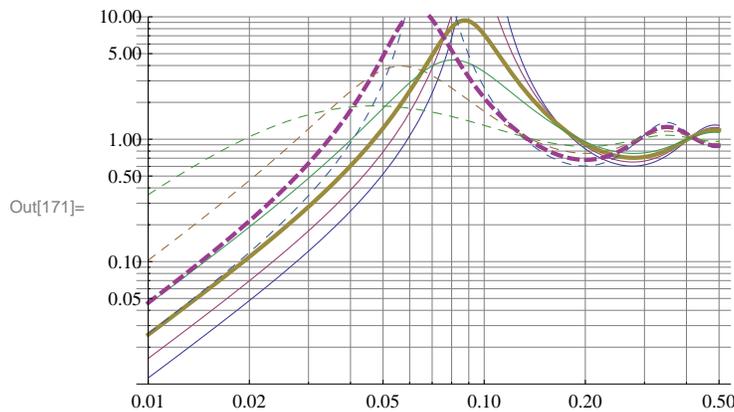
## ■ Closed loop transfer function curves

$$\text{In[95]:= } \gamma = .$$

```

In[171]:= LogLogPlot [ {
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]^2 /. {  $\alpha \rightarrow 2, \gamma \rightarrow .6$  },
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]^2 /. {  $\alpha \rightarrow 2, \gamma \rightarrow .5$  },
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]^2 /. {  $\alpha \rightarrow 2, \gamma \rightarrow .4$  },
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]^2 /. {  $\alpha \rightarrow 2, \gamma \rightarrow .3$  },
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]^2 /. {  $\alpha \rightarrow 3, \gamma \rightarrow .4$  },
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]^2 /. {  $\alpha \rightarrow 3, \gamma \rightarrow .3$  },
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]^2 /. {  $\alpha \rightarrow 3, \gamma \rightarrow .2$  },
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]^2 /. {  $\alpha \rightarrow 3, \gamma \rightarrow .1$  } },
  { f, .01, 1 / 2 }, GridLines -> Automatic, PlotRange -> { .01, 10 }, PlotStyle ->
  { Automatic, Automatic, Thick, Automatic, Dashed, { Thick, Dashed }, Dashed, Dashed } ]

```

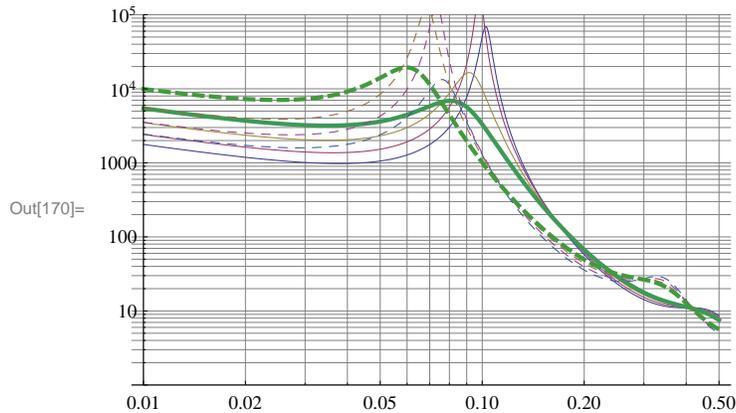


Closed loop transfer functions for various feedback gains (increasing crossover and resonant peak with increasing gain). The solid lines are for one sample time compute delay; dashed lines are two sample times of compute delay. Thick lines indicate optimal total-power-rejection curves for a Kolmogorov disturbance input spectrum.

```

In[170]:= LogLogPlot [ {
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]2 f-8/3 /. {α → 2, γ → .7},
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]2 f-8/3 /. {α → 2, γ → .6},
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]2 f-8/3 /. {α → 2, γ → .5},
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]2 f-8/3 /. {α → 2, γ → .4},
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]2 f-8/3 /. {α → 3, γ → .6},
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]2 f-8/3 /. {α → 3, γ → .5},
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]2 f-8/3 /. {α → 3, γ → .4},
  Abs [  $\frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2}$  ]2 f-8/3 /. {α → 3, γ → .3} },
  {f, .01, 1 / 2},
  GridLines → Automatic,
  PlotRange → {1, 100 000},
  PlotStyle →
  {Automatic, Automatic, Automatic, Thick, Dashed, Dashed, Dashed, {Thick, Dashed}} ]

```



Closed loop residual spectra for various feedback gains (increasing crossover and resonant peak with increasing gain). The solid lines are for one sample time compute delay; dashed lines are two sample times of compute delay. Thick lines indicate the least-total-integrated power curves.



$$\begin{aligned}
& \text{Abs} \left[ \frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2} \right]^2 f^{-8/3} /. \{ \alpha \rightarrow 3, \gamma \rightarrow .3 \}, \\
& \text{Abs} \left[ \frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2} \right]^2 f^{-8/3} /. \{ \alpha \rightarrow 3, \gamma \rightarrow .2 \} \}, \\
& \{ f, 10^{-6}, 1/2 \} + \\
& \left\{ \left\{ \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .6, \right. \right. \\
& \quad \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .5, \\
& \quad \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .4, \\
& \quad \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .3, \\
& \quad \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .2, \\
& \quad \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .1, \\
& \quad \left. \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .01 \right\}, \\
& \left\{ \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .6, \right. \\
& \quad \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .5, \\
& \quad \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .4, \\
& \quad \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .3, \\
& \quad \left. \frac{12 \pi^2 f_1^{1/3}}{\gamma^2} /. \gamma \rightarrow .2 \right\} /. f_1 \rightarrow 10^{-6} \\
& \left. \right) // \text{Re}
\end{aligned}$$

Out[168]= {{268.782, 108.9, 98.0197, 119.002, 192.586, 531.724, 22505.},  
{73.6983, 238.783, 279.958, 171.938, 224.203}}

$$\text{In[142]:= Series} \left[ \left( \frac{1}{1 + \frac{\gamma}{1 - e^{-i 2 \pi \alpha f}} e^{-i 2 \pi \alpha f} \left( \frac{1 - e^{-i 2 \pi f}}{-i 2 \pi f} \right)^2} \right) \left( \frac{1}{1 + \frac{\gamma}{1 - e^{i 2 \pi \alpha f}} e^{i 2 \pi \alpha f} \left( \frac{1 - e^{i 2 \pi f}}{i 2 \pi f} \right)^2} \right) f^{-8/3}, \{f, 0, 3\} \right]$$

$$\text{Out[142]=} \frac{4 \pi^2}{\gamma^2 f^{2/3}} + \frac{4 \left( -12 \pi^4 + 12 \pi^4 \gamma + 24 \pi^4 \alpha \gamma + \pi^4 \gamma^2 \right) f^{4/3}}{3 \gamma^4} + O[f]^{10/3}$$

$$\text{In[143]:= Integrate} \left[ \frac{4 \pi^2}{\gamma^2 f^{2/3}}, \{f, 0, f_1\} \right]$$

$$\text{Out[143]=} \frac{12 \pi^2 f_1^{1/3}}{\gamma^2}$$